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Seventh Quarterly Report

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For

NASA General Purpose Structural Analysis Program (NASTRAN)

(1 January 1968 to 1 April 1968)

Contract No. NAS5-10049

Prepared by

Computer Sciences Corporation
650 N. Sepulveda Boulevard
El Segundo, California 90245

For

Goddard Space Flight Center
Greenbelt, Maryland

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SUMMARY

During the seventh quarter effort centered on conversion of the NASTRAN statics program to IBM S/360 and UNIVAC 1108; work on the Statics Training Manual and the demonstration problems for the statics delivery; and continued implementation and testing of "statics remnants" and dynamics rigid formats and functional modules by the displacement method.

Section II discusses differential stiffness and buckling by the force method.

I. INTRODUCTION

The objective of contract NAS5-10049 is to develop the General Purpose Structural Analysis Program (NASTRAN). This program has the capability of solving static and dynamic structural problems by both the force and displacement methods via a problem-oriented format. It also has the capability of solving general matrix problems via a Direct Matrix Abstraction Package. The program will be delivered and installed at each of the participating eight NASA centers.

Computer Sciences Corporation is prime contractor and its two principal subcontractors include the MacNeal-Schwendler Corporation and the Martin-Marietta Corporation, Martin Company, Baltimore Division.

MacNeal-Schwendler is generally responsible for providing mathematical formulation of static analysis by the displacement method, real eigenvalue analysis and dynamic analysis. Additionally, MacNeal-Schwendler is providing consultation to the Martin Company in the development of static analysis by the force method; technical direction in the development of subroutines for decomposition of matrices; and analysis of test problems.

Martin Company is providing mathematical formulation and implementation of static analysis by the force method. Additionally, Martin is responsible for development of a structures plotting routine, and a routine to solve for the eigenvalues and eigenvectors of a real symmetric matrix by the tridiagonalization method.

The NASTRAN computer program itself is logically divided into four areas:

(1) Executive System;

(2) Structural Modules;

(3) Matrix Operations Routines; and

(4) Utility Routines.

The Executive System is responsible for:

(1) Control of the sequence of module executions according to options specified by the user;

(2) Communication of parameters to each module;

(3) Allocation of data sets (files) required during module execution; and

(4) Full recovery capability for restarting a program execution after either a scheduled or unscheduled interruption.

Structural Modules perform the actual problem solution. The Matrix Operation Routines provide a complete set of operations to be performed on matrices (e.g., addition, multiplication, decomposition, transposition, eigenvalue extraction, etc.). The Utility Routines provide a wide variety of common functions such as input and output of data sets, packing and unpacking of matrices, etc.

Work on the Statics Training Manual resulted in the following documents:

MS-38	NASTRAN Mnemonic Dictionary (revised)
MS-39	Statics Training Manual Outline
FMMD-3	Functional Module Mathematical Description for the Structural Matrix Assembly Modules
FMMD-4	Functional Module Mathematical Description for Stress and Data Recovery Modules (SDR1 and SDR2) for Statics and Normal Modes Analysis
FMMD-5	Structural Element Mathematical Description for the Displacement Method

FMMD-6	Structural Element Mathematical Descriptions for the Matrix Reduction Modules (MCE, SCE, SMP, RBMG, GPSP)
FMMD-7	Functional Module Mathematical Description for the Static Solution Generator Modules (SSG1, SSG2, SSG3, SSG4)
FMMS-59	Algorithm for BAR Element Stresses.

Work on the dynamics part of NASTRAN resulted in the completion of the following rigid formats for the Displacement Method:

PADF-21	Rigid Format Displacement Method - Direct Formulation Complex Eigenvalue Analysis
PADF-22	Rigid Format Displacement Method - Direct Formulation Frequency and Random Response
PADF-23	Rigid Format Displacement Method - Direct Formulation Transient Response
PADF-24	Rigid Format Displacement Method - Modal Formulation Complex Eigenvalue Analysis
PADF-25	Rigid Format Displacement Method - Modal Formulation Frequency and Random Response

PADF-26 Rigid Format
Displacement Method - Modal Formulation
Transient Response

The following technical memorandum was produced:

MS-14 Random Analysis Module

Work on the force method produced the following technical memoranda:

MBMD-2 Mathematical Description of Quadrilateral Super Element

MBMD-3 Mathematical Description of Stand Alone Shear and Twist Elements.

MBMD-4 Mathematical Description of Reaction Elements

MBMD-5 Mathematical Description of Structure Cutter

MBMD-6 Mathematical Description of Equation of Equilibrium Selector

Statics Training Manual - Outline of Structural Theory

Structural Modeling Techniques for Force Method

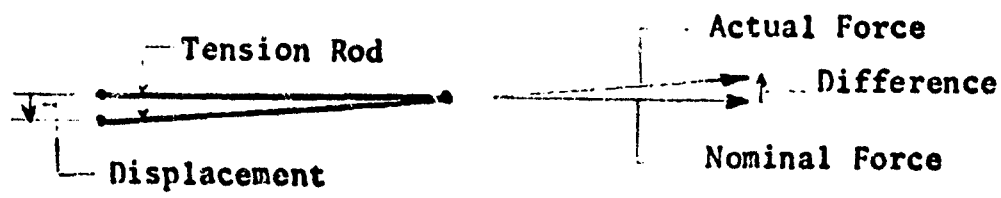
MBIDCF-11 Revised Input Card Descriptions for Force Method (for Training course).

II. DIFFERENTIAL STIFFNESS AND BUCKLING BY THE FORCE METHOD

A. INTRODUCTION

The following discussion considers two phenomena:

1. Change in Direction of Loaded Force Elements



2. Character of External Forces

When a structure deflects, gravity forces generally move with it, and, if the structure and its basic coordinate system are attached to the Earth (or other gravitating body), gravity forces remain fixed in direction. Some forces, such as rocket thrust and pressure loads, rotate with the deflected structure, and pressure loads also vary with expanding and contracting surface areas. The range of loading cases that can be routinely treated is limited.

In developing this section, two main areas are considered: the incorporation of differential stiffness in the analysis of structures (B.), and the compilation of differential stiffness formulas for force elements (C.). In addition, a Section D. considers treatment of non-gravity loads.

B. ANALYSIS OF STRUCTURES

1. Introduction

The differential stiffness of an element is the product of geometrical parameters (Section C.) and the internal load in the element. If the load in the element depended on its differential stiffness (to an appreciable extent) a non-linear problem would result. Formal solution of this problem is not provided in the force method. There are, however, structures in which differential stiffness is produced mainly by certain internal loads while its effect is felt mainly by another set. The most familiar example is the "beam-column". A limiting

case is a structure which may "buckle". In this case the "secondary" internal loads are initially absent, and can appear spontaneously, and sometimes catastrophically, when the primary loads reach a critical value.

The assembly of a differential stiffness matrix is as follows.

A set of internal loads, on which the differential stiffness of the elements is to be based, must be available, from a preliminary solution or estimate. Then

$$\{\Delta F\} = K_{bb}\{u\} \tag{1}$$

where forces $\{\Delta F\}$ and conjugate displacements $\{u\}$ are at grid points, in the basic coordinate directions, and K_{bb} are calculated by the formulas of Section C. The forces $\{\Delta F\}$ are those acting on the element.

Equation 1. can be transferred to a degree of freedom system which must be orthogonal at any grid point and must, of course, be sufficiently comprehensive.

$$\{P_g\} = K_b\{u_g\} \tag{2}$$

where $K_b = T_g{}^T K_{bb} T_g + K_{bc}$.

K_{bc} permits introduction of other sources of stiffness by direct input (see Section D.). K_b will be symmetrical unless K_{bc} is unsymmetrical.

To permit reference to cases where a structure is analyzed for more than one primary loading, K_b may be designated K_{bn}.

2. Ordinary Case – Supported Structure

It is advantageous, particularly for dynamic analysis*, to resolve loads into a set associated with degrees of freedom $\{P_g\}$, assumed not to contribute to

*(Note that differential stiffness as a function of time cannot be considered).

differential stiffness, and all others, such as generalized and thermal loads $\{P_{cn}\}$. Let $\{u_n\}$ and $\{f_n\}$ be the uncorrected deflections and internal loads from a loading P_{cn}. The differential stiffness matrix K_{bn} is based on the internal loads $\{f_n\}$. Then

$$\{u_g\} = A_g(-K_{bn}\{u_g\} + \{P_g\}) + \{u_n\} \tag{3}$$

whence

$$\{u_g\} = A_{gn}\{P_g\} + (I + A_g K_{bn})^{-1}\{u_n\} \tag{4}$$

where $A_{gn} = (I + A_g K_{bn})^{-1} A_g$.

Also

$$\{f\} = A_g(\{P_g\} - K_{bn}\{u_g\}) + \{f_n\} \tag{5}$$

whence

$$\{f\} = A_{gn}\{P_g\} - A_g K_{bn}(I + A_g K_{bn})^{-1}\{u_n\} + \{f_n\} \tag{6}$$

where $A_{gn} = A_g(I - K_{bn} A_{gn})$.

While, in principle, K_{bn} in Equation 3 to 6 need not be symmetrical, the users manual does apply restrictions.

3. Structure with Rigid Body Degrees of Freedom

A simple procedure, when applicable, is to perform a differential stiffness analysis, as in Section 2, on a structure already freed by inertia relief. This is an exact procedure when all external loads are already in equilibrium prior to inertia relief. (In fact, the fictitious reactions vanish anyway in this case.) Generally, however, it will not give correct results for other loads.

4. Buckling

Any buckling problem which can be completely described by a differential stiffness matrix can be stated as follows.

$$(I - \frac{1}{\lambda} AK_h)\{u_h\} = \{0\} \tag{7}$$

where K_h is the differential stiffness matrix for a unit value of the unknown buckling load P_{cr}, and A may be the elastic deflection matrix A_g, but may also be modified by the differential stiffness from other load held constant while P_{cr} varies.

Equation 7 is solved for the eigenvalues

$$(-1/\lambda) = P_{cr} \tag{8}$$

and for the mode shapes given by the eigenvectors $\{u_h\}$. If K_h is not symmetrical appropriate routines are required and there may be some imaginary or complex answers, and furthermore, since an unsymmetrical K_h implies nonconservative loading, dynamic instability may invalidate real roots as well.

C. FORMULAS FOR ELEMENTS

1. Introduction

This section presents formulas for K_{bb} (Equation 1.) for various force elements, and, equally important, the basis for them and the limitations thereby implied.

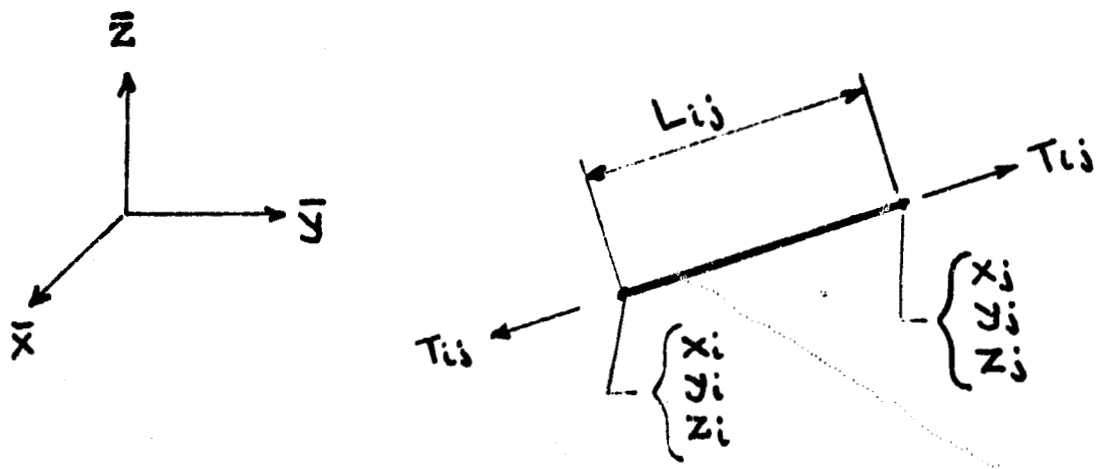
Not all force elements are equally important. In most cases, the effects of differential stiffness, buckling for instance, are associated with tension, compression and shear. Beyond this, about the only significant phenomenon that occurs is the buckling of deep narrow beams under the action of bending

moment. In succeeding subsections some minor terms are developed but not retained for programming.

Expedients for deriving formulas are presented as the need arises.

2. Simple Tension Element

The formula will be stated and then derived.



$$\left\{\begin{array}{c} \Delta F_i^x \\ \Delta F_i^y \\ \Delta F_i^z \\ \hline \Delta F_j^x \\ \Delta F_j^y \\ \Delta F_j^z \end{array}\right\} = K_{bb} \left\{\begin{array}{c} u_i^x \\ u_i^y \\ \cdot \\ \hline \cdot \\ \cdot \\ u_j^z \end{array}\right\} \tag{9}$$

where

$$K_{bb} = \frac{T_{ij}}{L_{ij}} \left[\begin{array}{c|c} \alpha & -\alpha \\ \hline -\alpha & \alpha \end{array}\right] \tag{10}$$

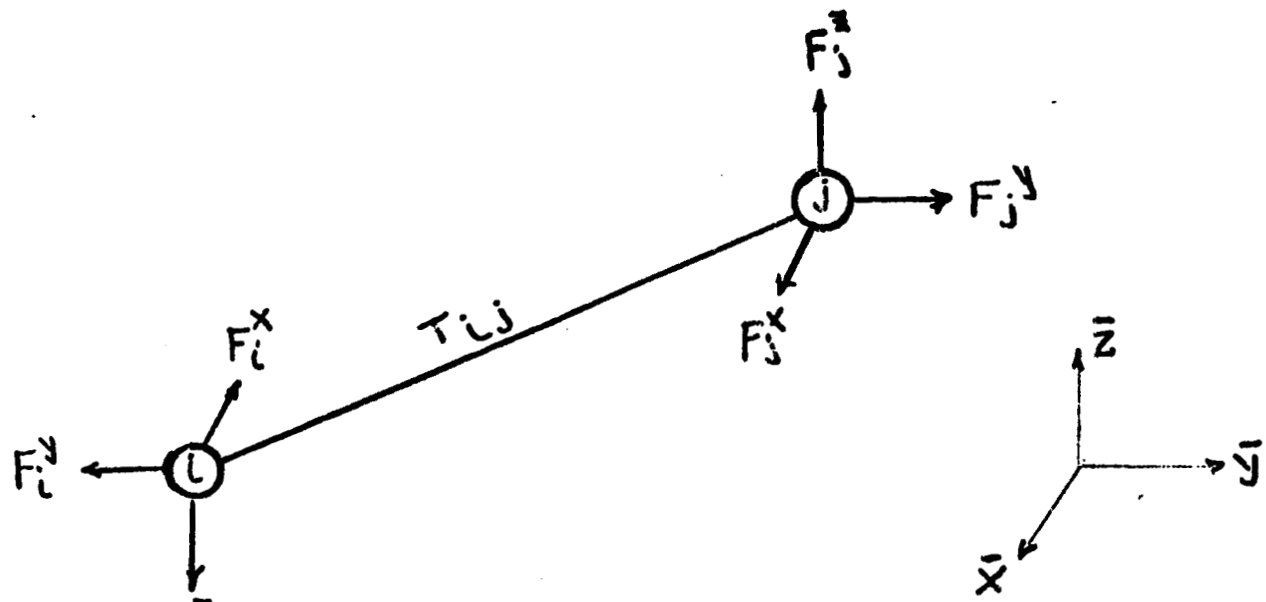
and

$$\alpha = I_{3x3} - \left\{\begin{array}{c} a_r \\ b_r \\ c_r \end{array}\right\} \lfloor a_r b_r c_r \rfloor$$

and a_r, b_r, c_r are projections of the unit vector $\bar{r}$ on the x-, y-, and z-axes.

a. Derivation

A very straightforward procedure directly applicable to tension elements (and more or less directly applicable to membrane elements) is to differentiate the equilibrium equations.



(Forces acting on ends of bar)

$$\begin{Bmatrix} F_i^x \\ F_i^y \\ F_i^z \\ F_j^x \\ F_j^y \\ F_j^z \end{Bmatrix} = T_{ij} \begin{Bmatrix} (x_i-x_j)/L \\ (y_i-y_j)/L \\ (z_i-z_j)/L \\ (x_j-x_i)/L \\ (y_j-y_i)/L \\ (z_j-z_i)/L \end{Bmatrix} \tag{11}$$

$$L = \sqrt{(x_j-x_i)^2 + (y_j-y_i)^2 + (z_j-z_i)^2}$$

We wish to construct a 6x6 differential stiffness matrix $K_{bb}(T_{ij})$, as in Equation 9.

The displacements are increments to coordinates. For instance $u_i^x = \Delta x_i$. Suppose we wish to construct the first term (col. 1, row 1) of K_{bb}

$$F_i^x = T_{ij} \frac{x_i - x_j}{L}$$

$$\frac{d}{dx_i} F_i^x = T_{ij}\left(\frac{1}{L} - \frac{x_i - x_j}{L^2} \frac{dL}{dx_i}\right)$$

and

$$\frac{dL}{dx_i} = \frac{x_i - x_j}{L}$$

Whence

$$\Delta F_i^x = T_{ij} \left(\frac{1}{L} - \frac{(x_i - x_j)^2}{L^3}\right) u_i^x \tag{12}$$

$$\Delta F_i^x \quad T_{ij} \frac{L^2 - (x_i - x_j)^2}{L^3} u_i^x$$

For col 1, row 2

$$\frac{d}{dx_i} F_i^y = T_{ij}\left(- \frac{(y_i - y_j)(x_i - x_j)}{L^3}\right) \tag{13}$$

By simple permutations, all the other terms follow and the results can be condensed by calling the direction cosines of $\overline{L}$ a, b, and c.

$$(a = (x_j - x_i)/L, \text{ etc.})$$

$$K_{bb}(T_{ij}) = \frac{T_{ij}}{L} \left[\begin{array}{ccc|cc} 1-a^2 & -ab & -ac & -1+a^2 & \cdots \\ -ab & 1-b^2 & -bc & ab & \cdots \\ -ac & -bc & 1-c^2 & ac & \cdots \\ \hline -1+a^2 & ab & ac & \cdots & \text{etc.} \\ ab & -1+b^2 & bc & \cdots & \\ ac & bc & -1+c^2 & \cdots & \end{array}\right] \qquad (14)$$

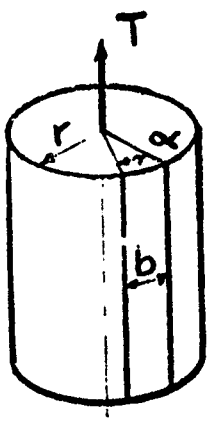
With but little further manipulation Equation 10 results.

3. Quadrilateral Tension Element
Triangular Tension Element

The tension load in a plate element can be expressed as

$$F = N\,b \qquad (15)$$

where N is a force per unit width and b is the width. It cannot be said universally whether a variation of b causes N or F or both to vary. Consider a sector of a uniformly end loaded cylinder.



$$b = r\alpha$$

$$N = \frac{T}{2\pi r}$$

$$F = \frac{T\alpha}{2\pi}$$

Under a simple load T, F is constant, N varies inversely with r. Under a uniform pressure, $T = \pi r^2 q$, F varies with r^2 and N varies with r.

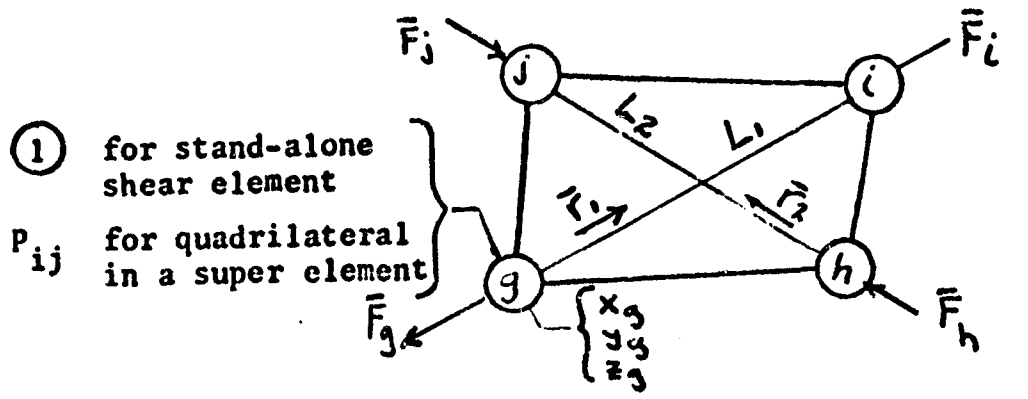
For the differential stiffness of the element, F will be considered constant, leaving variation of F to be treated, if at all, as an external load differential stiffness (Section D). This will make the formulation identical to that for tension bar elements, N b replacing T_{ij}.

$$K_{bb} = \frac{N\,b}{L}\begin{bmatrix} a & -a \\ -a & a \end{bmatrix} \tag{16}$$

4. Shear Element

The phenomena that take place when the four corners of a quadrilateral are displaced are far too complicated to be described very well by a shear element and four tension elements. The procedure which will be followed will be to differentiate the equilibrium equations, holding the shear flow (q for stand-alone, N^{IJ} for super element) constant.
While this may not appear consistent with the membrane tension element, where N b was held constant, it does provide a simple method that correctly accounts for all major effects.

Derivations and formulas are given for "pure shear" elements (used in the program) and "tangentially loaded" elements (used in later development, Section 5).



① for stand-alone shear element

P_{ij} for quadrilateral in a super element

(17)

$$L_1 = \sqrt{(x_i-x_g)^2 + (y_i-y_g)^2 + (z_i-z_g)^2}$$

Direction cosines a_1, b_1, c_1

$$L_2 = \sqrt{(x_j-x_h)^2 + (y_j-y_h)^2 + (z_j-z_h)^2}$$

Direction cosines a_2, b_2, c_2

a. Pure Shear Element

$$\begin{Bmatrix} F_g^x \\ F_g^y \\ F_g^z \\ \hline F_h^x \\ F_h^y \\ F_h^z \\ \hline F_i^x \\ F_i^y \\ F_i^z \\ \hline F_j^x \\ F_j^y \\ F_j^z \end{Bmatrix} = \frac{q}{2} \begin{Bmatrix} & a_1 \\ -L_2 & b_1 \\ & c_1 \\ \hline & a_2 \\ L_1 & b_2 \\ & c_2 \\ \hline & a_1 \\ L_2 & b_1 \\ & c_1 \\ \hline & a_2 \\ -L_1 & b_2 \\ & c_2 \end{Bmatrix} \tag{18}$$

Typical derivatives are

$$\frac{dL_1}{dx_g} = -a_1 \; ; \quad \frac{dL_2}{dx_g} = 0 \quad (dx_g = u_g^x)$$

$$\frac{da_1}{dx_g} = \frac{d}{dx_g}\frac{x_i - x_g}{L_1} = -\frac{1}{L_1}(1-a_1^2)$$

$$\frac{db_1}{dx_g} = \frac{a_1 b_1}{L_1} \; ; \quad \frac{da_2}{dx_g} = 0$$

Then

$$K_{bb} = \frac{q}{2}\begin{bmatrix} -\beta' & \vdots & -\beta' \\ -\beta & \vdots & \beta \end{bmatrix} \tag{19}$$

where

$$\beta' = \begin{bmatrix} \frac{L_2}{L_1}(1-a_1^2) & -\frac{L_2}{L_1}a_1b_1 & -\frac{L_2}{L_1}a_1c_1 & -a_1a_2 & -a_1b_2 & -a_1c_2 \\ & \frac{L_2}{L_1}(1-b_1^2) & -\frac{L_2}{L_1}b_1c_1 & -b_1a_2 & -b_1b_2 & -b_1c_2 \\ & & \frac{L_2}{L_1}(1-c_1^2) & -c_1a_2 & -c_1b_2 & -c_1c_2 \\ & & & \frac{L_1}{L_2}(1-a_2^2) & -\frac{L_1}{L_2}a_2b_2 & -\frac{L_1}{L_2}a_2c_2 \\ & & \text{Sym.} & & \frac{L_1}{L_2}(1-b_2^2) & -\frac{L_1}{L_2}b_2c_2 \\ & & & & & \frac{L_1}{L_2}(1-c_2^2) \end{bmatrix}$$

With a little further manipulation the differential stiffness formulas for the pure shear element can be reduced to the following form.

$$\begin{Bmatrix} F_g^x \\ F_g^y \\ F_g^z \\ F_h^x \\ \cdot \\ \cdot \\ \cdot \\ F_j^z \end{Bmatrix} = K_{bb}{}_{(12x12)} \begin{Bmatrix} u_g^x \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_j^z \end{Bmatrix} \qquad (20)$$

where

$$K_{bb} = \frac{q}{2L_1L_2} \begin{bmatrix} \beta & -\beta \\ -\beta & \beta \end{bmatrix}$$

β is the following 6 x 6 matrix

$$\beta = \left[(L_2)^2 \quad (L_2)^2 \quad (L_2)^2 \quad (L_1)^2 \quad (L_1)^2 \quad (L_1)^2 \right] - \gamma^T\gamma$$

and

$$\gamma = \left[L_2a_1 \quad L_2b_1 \quad L_2c_1 \quad L_1a_2 \quad L_1b_2 \quad L_1c_2 \right]$$

b. Tangentially-Loaded Shear Element

This model turns out to be especially simple to treat, vectorially.

$$\begin{Bmatrix} \bar{F}_g \\ \bar{F}_h \\ \bar{F}_i \\ \bar{F}_j \end{Bmatrix} = \frac{q}{2} \begin{Bmatrix} -\bar{L}_1 \\ \bar{L}_2 \\ \bar{L}_1 \\ -\bar{L}_2 \end{Bmatrix} \tag{21}$$

The differential relationships are

$$\Delta\bar{L}_1 = \bar{u}_i - \bar{u}_g$$
$$\Delta\bar{L}_2 = \bar{u}_j - \bar{u}_h$$

Whence

$$\begin{Bmatrix} \Delta\bar{F}_g \\ \Delta\bar{F}_h \\ \Delta\bar{F}_i \\ \Delta\bar{F}_j \end{Bmatrix} = \frac{q}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \bar{u}_g \\ \bar{u}_h \\ \bar{u}_i \\ \bar{u}_j \end{Bmatrix} \tag{22}$$

(Equation 22 is used later in developing beam formulas)

5. Beams (Unpaired)

The fundamental elements of structural models are tension and shear elements. A beam is a special bonus provided to expedite the representation of long narrow components whose behavior can be idealized without excessive loss of

accuracy. Accounting for various degrees of complexity (such as short fatness) is a never-ending game.

The differential stiffness of a beam is likewise a kind of bonus, encountered in such specialties as torsion buckling of columns and lateral buckling of beams, and is rather complicated.

The formulas are developed for prismatic beams with doubly symmetric cross-section.

The development used in this section employs a tubular model of a beam, the model consisting of two rigid end bulkheads, four edge tension elements, and four rectangular tangentially loaded shear elements (= 8 diagonals).

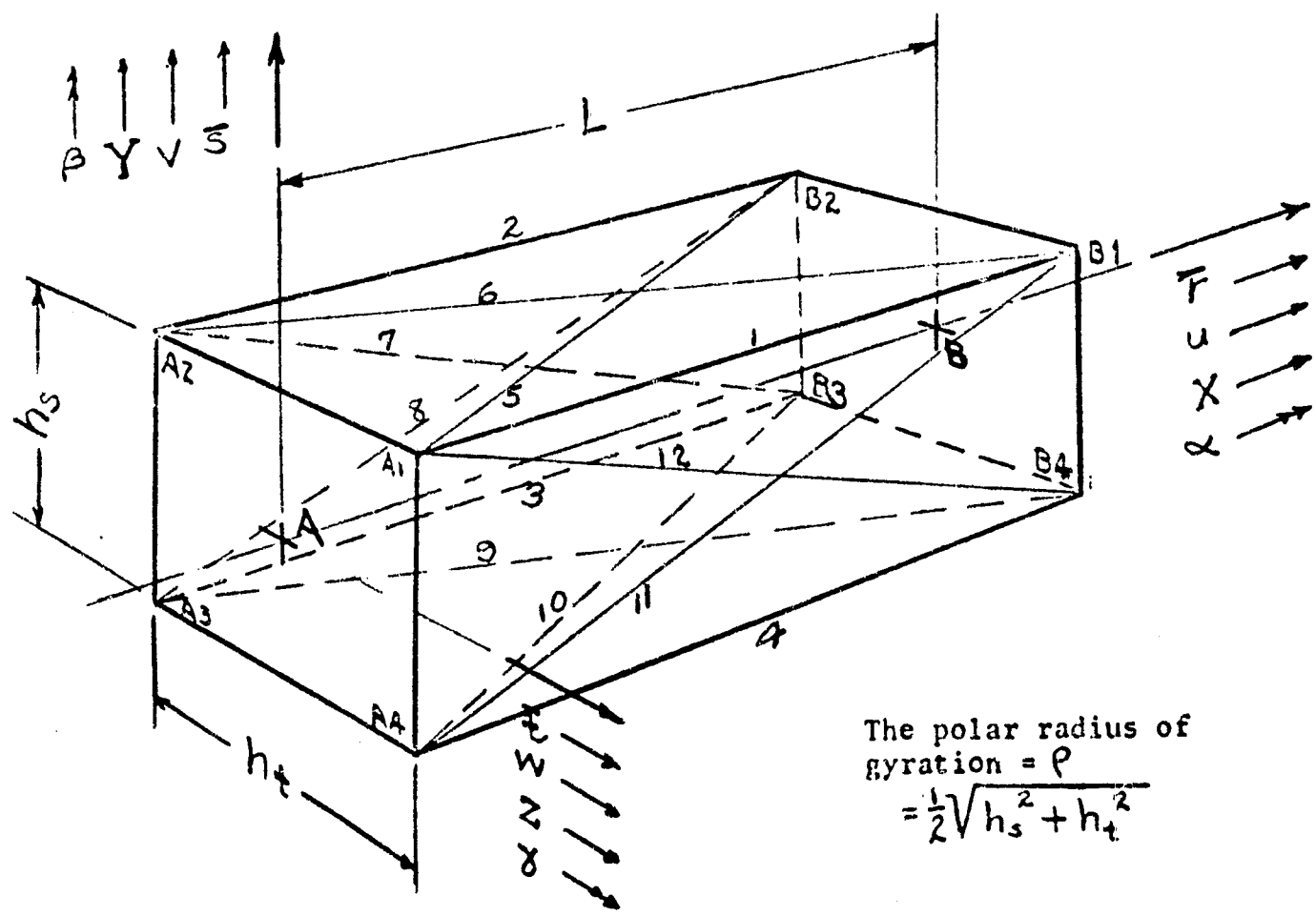
This model reproduces most of the properties used in the displacement method but differs in some respects.

(1) The development suggests how to preserve symmetry in the differential stiffness matrix for a torsion rod.

(2) The restriction to double symmetry is explained.

(3) Some integrals are missing and some trivial terms are developed (but discarded).

The model is as follows.



The polar radius of gyration = ρ

$$= \frac{1}{2}\sqrt{h_s^2 + h_t^2}$$

The beam represented by this model is as follows:

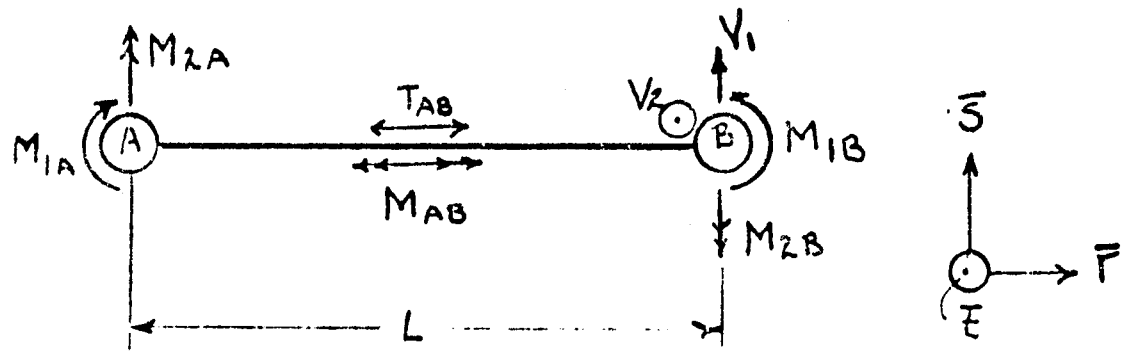


TABLE 1

MODEL ELEMENT LOADS VS. BEAM LOADS

Elem	T_{AB}	M_{AB}	M_{2A}	M_{2B}	M_{1A}	M_{1B}
1	1/4	0	$-1/4h_t$	$-1/4h_t$	$-1/4h_s$	$-1/4h_s$
2	1/4	0	$+1/4h_t$	$+1/4h_t$	$-1/4h_s$	$-1/4h_s$
3	1/4	0	$+1/4h_t$	$+1/4h_t$	$+1/4h_s$	$+1/4h_s$
4	1/4	0	$-1/4h_t$	$-1/4h_t$	$+1/4h_s$	$+1/4h_s$
5	0	$-\frac{L}{4h_s h_t}$	$-1/4h_t$	$+1/4h_t$	0	0
6	0	$+\frac{L}{4h_s h_t}$	$+1/4h_t$	$-1/4h_t$	0	0
7	0	$-\frac{L}{4h_s h_t}$	0	0	$-1/4h_s$	$+1/4h_s$
8	0	$+\frac{L}{4h_s h_t}$	0	0	$+1/4h_s$	$-1/4h_s$
9	0	$-\frac{L}{4h_s h_t}$	$+1/4h_t$	$-1/4h_t$	0	0
10	0	$+\frac{L}{4h_s h_t}$	$-1/4h_t$	$+1/4h_t$	0	0
11	0	$-\frac{L}{4h_s h_t}$	0	0	$+1/4h_s$	$-1/4h_s$
12	0	$+\frac{L}{4h_s h_t}$	0	0	$-1/4h_s$	$+1/4h_s$

Note: r-components of loads in diagonals are given.
(Note that Tension/Length = r-component/(L).)

Differential stiffness will be derived for one beam loading at a time, but in the case of bending will be developed separately for the chord loads, which, in this model, depend on $M_A + M_B$, and for the diagonal loads, which depend on

$$V_2 = (M_{2A} - M_{2B})/L$$

and (23)

$$V_1 = (M_{1A} - M_{1B})/L$$

The differential stiffness matrices for the components are as follows.

Tension Bars: Since these coincide with the x-axis, their equation can be simpler than Equation 10 and 11.

$$\begin{Bmatrix} \Delta Y_A \\ \Delta Y_B \\ \Delta Z_A \\ \Delta Z_B \end{Bmatrix} = K_{AB} \begin{Bmatrix} v_A \\ v_B \\ w_A \\ w_B \end{Bmatrix} \quad (24)$$

where

$$K_{AB} = \frac{T}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Shear Elements: Use Equation 22. (Note that in this model the shear panels can, in fact, only be loaded tangentially.)

a. Derivations

(1) Beam Loaded by Tension T_{AB}

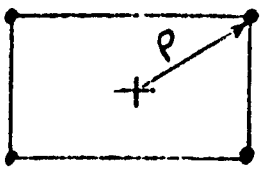
Only bars 1-4 are loaded.

(a) Response to Lateral Displacement

Since $v_{A1} = v_{A2} = v_{A3} = v_{A4} = v_A$, etc., this model gives an equation identical to Equation 24 except that T_{AB} replaces T.

There is no response to displacements u or rotations β or γ .

(b) Response to Rotations α



Tangential displacement at corners is equal to $\rho\alpha$ (with appropriate signs).

$$\begin{Bmatrix} \Delta M_{Ar} \\ \Delta M_{Br} \end{Bmatrix} = \frac{T_{AB}\rho^2}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_A \\ \alpha_B \end{Bmatrix} \tag{25}$$

(This is the driving force for torsion buckling of columns with large polar moment of inertia and small torsional stiffness).

Equation 24 (as modified) and 25 are the only ones containing T_{AB}.

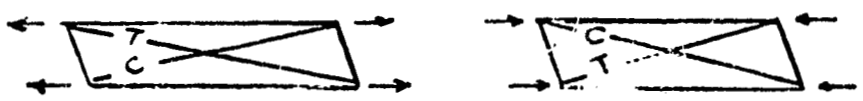
(b) Beam Loaded by Torsion M_{AB}

The least bending stiffness of a thin "blade" is typically a little more than one half its torsional stiffness. Such a beam could buckle under a torsional moment since it would rather curl up ( ) than twist ( ). Since this is a rather special phenomenon, and, as it turns out, particularly finicky about external loading, no provision is made for it in the program. For the record, the formulas developed via the tubular model are included here.

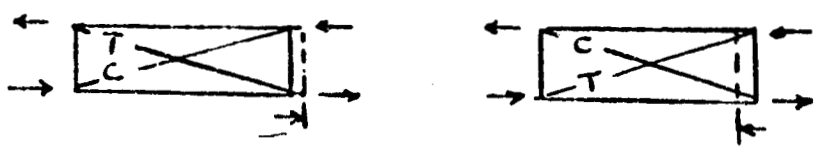
We observe first that there is no response to displacements v or w or rotations α: Any displacement v of an edge such as A_1A_4 produces two equal opposing forces at each end of face 1-4 (in diagonals 11 and 12) and at A_1A_4 and B_2B_3 (in diagonals 5 and 10), etc. There is no response to elongation ($u_{A1} = u_{A2} = u_{A3} = u_{A4}$, etc.) because at each corner there are two oppositely loaded diagonals.

(a) Response to End Rotations β and γ

Notice that identical rotations at both ends translate one pair of faces and transform the other pair into parellelograms:



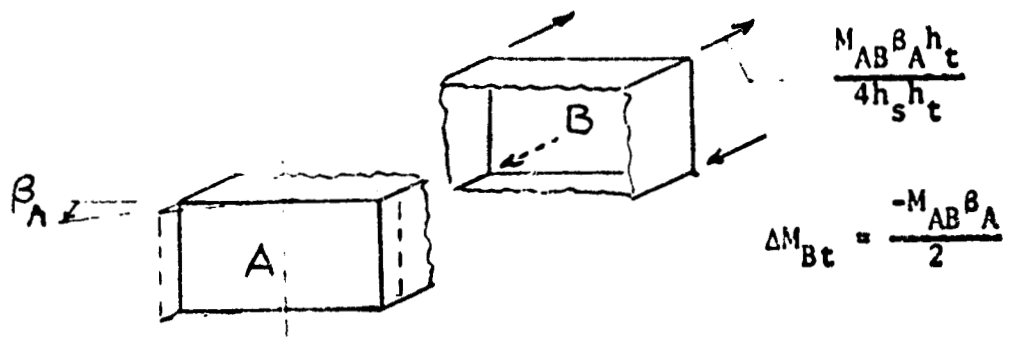
while equal opposite roations transform one pair into trapezoids (= translation of diagonals; no response) while stretching or shortening the other pair.



Consider a rotation β_A

$$u_{A1} = - u_{A2} = - u_{A3} = u_{A4} = \beta_A h_t/2$$

At each of the corners $A_1 \ldots A_4$, the forces cancel but at B they are additive.



Similarly we calculate three other cases and get

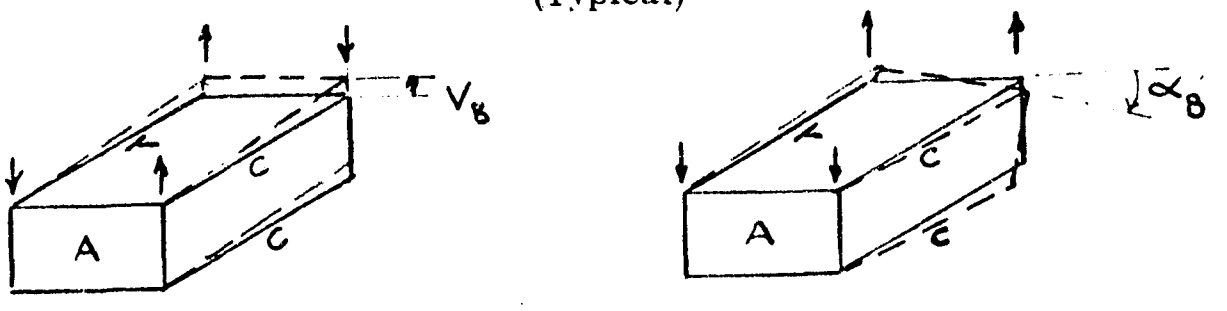
$$\begin{Bmatrix} \Delta M_A s \\ \Delta M_B s \\ \Delta M_A t \\ \Delta M_B t \end{Bmatrix} = \frac{M_{AB}}{2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \beta_A \\ \beta_B \\ \gamma_A \\ \gamma_B \end{Bmatrix} \qquad (26)$$

Equation 26 is found to have the following properties. (1) It is symmetrical. (2) On the average it gives a correct representation of the moment change along a constantly curving beam ←·→ . (3) The following end conditions are implied: One half rotates with the end of the beam. (This is because in the model the uniformly distributed forces producing M_{AB} moved with the model but did not change direction.)

(c) Beam Loaded by Symmetrical Bending Moments

Equations are derived for $M_{2A} + M_{2B}$ (typical). There are only chord loads $-T_1 = +T_2 = +T_3 = -T_4 = (1/4h_t)(M_{2A} + M_{2B})$. This system is wholly unresponsive to u displacements. Forces from w_A and w_B are self cancelling. There remain displacements v and v-components of rotations α.

(Typical)



$$\begin{Bmatrix} \Delta Y_A \\ \Delta Y_B \\ \Delta M_{Ar} \\ \Delta M_{Br} \end{Bmatrix} = \frac{M_{2A}+M_{2B}}{2L} \begin{bmatrix} 0 & 0 & +1 & -1 \\ 0 & 0 & -1 & +1 \\ +1 & -1 & 0 & 0 \\ -1 & +1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_A \\ v_B \\ \alpha_A \\ \alpha_B \end{Bmatrix} \tag{27}$$

Similarly

$$\begin{Bmatrix} \Delta Z_A \\ \Delta Z_B \\ \Delta M_{Ar} \\ \Delta M_{Br} \end{Bmatrix} = \frac{M_{1A}+M_{1B}}{2L} \begin{bmatrix} 0 & 0 & -1 & +1 \\ 0 & 0 & +1 & -1 \\ -1 & +1 & 0 & 0 \\ +1 & -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} w_A \\ w_B \\ \alpha_A \\ \alpha_B \end{Bmatrix} \tag{28}$$

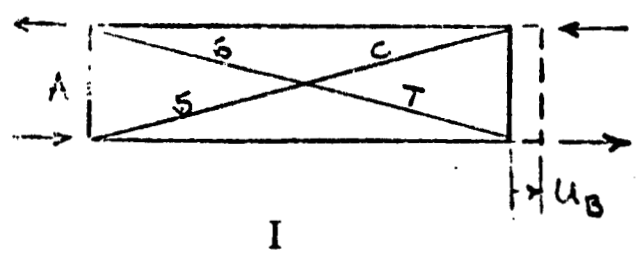
(d) Beam Loaded by Shear

Take first of Equations 23 as typical. The only element loads are

$-T_5 = T_6 = T_9 = -T_{10} = \frac{1}{4}\frac{L}{h_t} V_2$. Forces and displacements related to u,

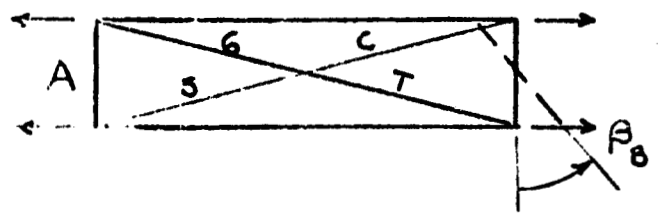
and those related to v form separate systems. (w-terms are self cancelling.)

Displacements u



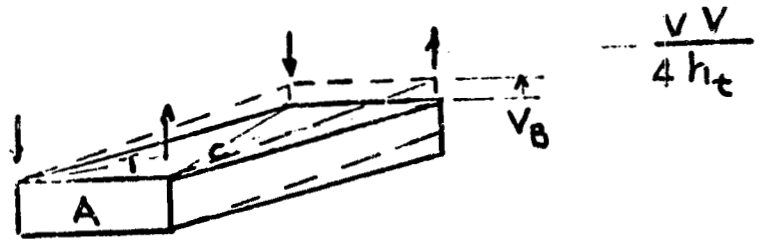
I

Rotations β

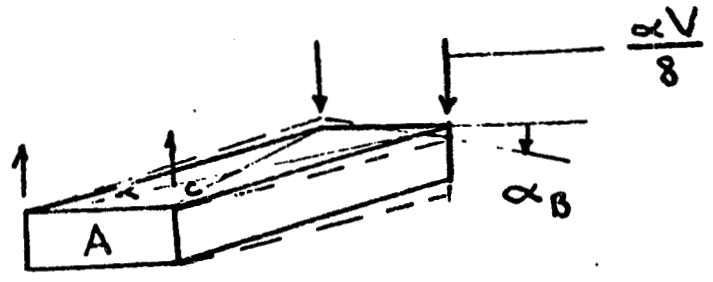


$$\begin{Bmatrix} \Delta X_A \\ \Delta X_B \\ \Delta M_{As} \\ \Delta M_{Bs} \end{Bmatrix} = \frac{M_{2A}-M_{2B}}{2L} \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & +1 & +1 \\ -1 & +1 & 0 & 0 \\ -1 & +1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_A \\ u_B \\ \beta_A \\ \beta_B \end{Bmatrix} \quad (29)$$

Displacements v



Rotations α



$$\begin{Bmatrix} \Delta Y_A \\ \Delta Y_B \\ \Delta M_{Ar} \\ \Delta M_{Br} \end{Bmatrix} = \frac{M_{2A}-M_{2B}}{2L} \begin{bmatrix} 0 & 0 & +1 & +1 \\ 0 & 0 & -1 & -1 \\ +1 & -1 & 0 & 0 \\ +1 & -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_A \\ v_B \\ \alpha_A \\ \alpha_B \end{Bmatrix} \tag{30}$$

Similarly

$$\begin{Bmatrix} \Delta X_A \\ \Delta X_B \\ \Delta M_{At} \\ \Delta M_{Bt} \end{Bmatrix} = \frac{M_{1A}-M_{1B}}{2L} \begin{bmatrix} 0 & 0 & +1 & +1 \\ 0 & 0 & -1 & -1 \\ +1 & -1 & 0 & 0 \\ +1 & -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_A \\ u_B \\ \gamma_A \\ \gamma_B \end{Bmatrix} \tag{31}$$

$$\begin{Bmatrix} \Delta Z_A \\ \Delta Z_B \\ \Delta M_{Ar} \\ \Delta M_{Br} \end{Bmatrix} = \frac{M_{1A}-M_{1B}}{2L} \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & +1 & +1 \\ -1 & +1 & 0 & 0 \\ -1 & +1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} w_A \\ w_B \\ \alpha_A \\ \alpha_B \end{Bmatrix} \tag{32}$$

Some of these terms may not be very important, but their physical interpretation is interesting. The increment of bending moment produced by extension goes with the relation Mom = Shear x Distance; the conjugate relation is simply

.

(e) Beam Loaded by Individual Bending Moments

Some of the results derived in Sections (d) and (e) can be recombined so as to be stated in terms of the individual moment M_{2A}, M_{2B}, M_{1A}, and M_{1B}.

$$\begin{Bmatrix} \Delta Y_A \\ \Delta Y_B \\ \Delta M_{Ar} \\ \Delta M_{Br} \end{Bmatrix} = \frac{1}{L} \begin{bmatrix} 0 & 0 & M_{2A} & -M_{2B} \\ 0 & 0 & -M_{2A} & M_{2B} \\ M_{2A} & -M_{2A} & 0 & 0 \\ -M_{2B} & M_{2B} & 0 & 0 \end{bmatrix} \begin{Bmatrix} v_A \\ v_B \\ \alpha_A \\ \alpha_B \end{Bmatrix} \tag{33}$$

$$\begin{Bmatrix} \Delta Z_A \\ \Delta Z_B \\ \Delta M_{Ar} \\ \Delta M_{Br} \end{Bmatrix} = \frac{1}{L} \begin{bmatrix} 0 & 0 & -M_{1A} & +M_{1B} \\ 0 & 0 & +M_{1A} & -M_{1A} \\ -M_{1A} & +M_{1A} & 0 & 0 \\ +M_{1B} & -M_{1B} & 0 & 0 \end{bmatrix} \begin{Bmatrix} w_A \\ w_B \\ \alpha_A \\ \alpha_B \end{Bmatrix} \tag{34}$$

b. Complete Differential Stiffness Matrix for Simple Beam Element

The various matrices developed in the preceding section (a.) can be assembled to form a symmetrical 12 x 12 matrix (Table 2). Only the principal terms, enclosed by a heavy line and forming a 6 x 6 matrix, are used in further development, however (and in a majority of applications only the 4 x 4 matrix relating displacements v_A, v_B, w_A, w_B to their conjugate forces is likely to be needed.)

It is possible to write equations to transfer forces and displacements to a pair of grid points not on the centroidal axis but it appears that they would serve no useful purpose, for the following reasons. (1) Differential stiffness of beams is significant only in isolated beams. (2) Transfer to eccentric grid points entails a full set of moment equations at each grid point, precluding the use of line super elements. Hence, in a general case, there need be no alternative to individual beam segments with grid points on their centroidal axes.

The differential stiffness matrix for a beam, based on the 6 x 6 matrix in Table 2 can be set up in terms of beam coordinates (K_{beam}) and in basic coordinates (K_{bb}). (The five internal loads producing differential stiffness are shown below.)

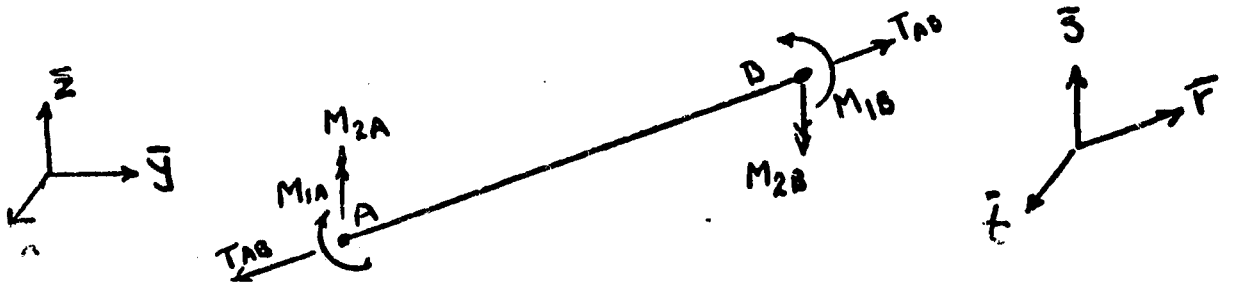


TABLE 2. DIFFERENTIAL STIFFNESS MATRIX FOR BEAM SEGMENT

	u_A	u_B	v_A	v_B	w_A	w_B	α_A	α_B	β_A	β_B	γ_A	γ_B
ΔX_A									$\frac{M_{2B}-M_{2A}}{2L}$	$\frac{M_{2B}-M_{2A}}{2L}$	$\frac{M_{1A}-M_{1B}}{2L}$	$\frac{M_{1A}-M_{1B}}{2L}$
ΔX_B									$\frac{M_{2A}-M_{2B}}{2L}$	$\frac{M_{2A}-M_{2B}}{2L}$	$\frac{M_{1B}-M_{1A}}{2L}$	$\frac{M_{1B}-M_{1A}}{2L}$
ΔY_A			$\frac{T_{AB}}{L}$	$\frac{-T_{AB}}{L}$			$\frac{M_{2A}}{L}$	$\frac{-M_{2B}}{L}$				
ΔY_B			$\frac{-T_{AB}}{L}$	$\frac{T_{AB}}{L}$			$\frac{-M_{2A}}{L}$	$\frac{M_{2B}}{L}$				
ΔZ_A					$\frac{T_{AB}}{L}$	$\frac{-T_{AB}}{L}$	$\frac{-M_{1A}}{L}$	$\frac{M_{1B}}{L}$				
ΔZ_B					$\frac{-T_{AB}}{L}$	$\frac{T_{AB}}{L}$	$\frac{M_{1A}}{L}$	$\frac{-M_{1B}}{L}$				
ΔM_{Ar}			$\frac{+M_{2A}}{L}$	$\frac{-M_{2A}}{L}$	$\frac{-M_{1A}}{L}$	$\frac{M_{1A}}{L}$	$\frac{T_{AB}\rho^2}{L}$	$\frac{-T_{AB}\rho^2}{L}$				
ΔM_{Br}			$\frac{-M_{2B}}{L}$	$\frac{+M_{2B}}{L}$	$\frac{M_{1B}}{L}$	$\frac{-M_{1B}}{L}$	$\frac{-T_{AB}\rho^2}{L}$	$\frac{T_{AB}\rho^2}{L}$				
ΔM_{As}	$\frac{M_{2B}-M_{2A}}{2L}$	$\frac{M_{2A}-M_{2B}}{2L}$										$\frac{-M_{AB}}{2}$
ΔM_{Bs}	$\frac{M_{2B}-M_{2A}}{2L}$	$\frac{M_{2A}-M_{2B}}{2L}$									$\frac{M_{AB}}{2}$	
ΔM_{At}	$\frac{M_{1A}-M_{1B}}{2L}$	$\frac{M_{1B}-M_{1A}}{2L}$								$\frac{M_{AB}}{2}$		
ΔM_{Bt}	$\frac{M_{1A}-M_{1B}}{2L}$	$\frac{M_{1B}-M_{1A}}{2L}$							$\frac{-M_{AB}}{2}$			

ρ = polar radius of gyration

In r, s, t coordinates

$$\{\Delta F_{beam}\} = K_{beam} \{u_{beam}\} \tag{35}$$

where

$$\{\Delta F_{beam}\} = \{\Delta F_A^s \quad \Delta F_B^s \quad \Delta F_A^t \quad \Delta F_B^t \quad \Delta M_A^r \quad \Delta M_B^r\}$$

$$\{u_{beam}\} = \{u_A^s \quad u_B^s \quad u_A^t \quad u_B^t \quad \theta_A^r \quad \theta_B^r\}$$

and

$$K_{beam} = \frac{1}{L}\begin{bmatrix} T_{AB} & -T_{AB} & 0 & 0 & M_{2A} & -M_{2B} \\ & T_{AB} & 0 & 0 & -M_{2A} & M_{2B} \\ & & T_{AB} & -T_{AB} & -M_{1A} & M_{1B} \\ & & & T_{AB} & M_{1A} & -M_{1B} \\ & & & & T_{AB}\rho^2 & -T_{AB}\rho^2 \\ & \text{Sym.} & & & & T_{AB}\rho^2 \end{bmatrix}$$

In basic coordinates

$$\{\Delta F\} = K_{bb} \{u\} \tag{36}$$

where

$$\{\Delta F\} = \{\Delta F_i^x \quad \Delta F_i^y \quad \Delta F_i^z \quad \Delta F_j^x \quad \Delta F_j^y \quad \Delta F_j^z \quad \Delta M_i^x \ \ldots\ldots \ \Delta M_j^z\}$$

$$\{u\} = \{u_i^x \quad u_i^y \ \ldots\ldots\ldots\ldots\ldots\ u_j^z \quad \theta_i^x \ \ldots\ldots \ \theta_j^z\}$$

$$K_{bb} = T_1^T K_{beam} T_1$$

and

$$T_1 = \begin{bmatrix} a_s & b_s & c_s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_s & b_s & c_s & 0 & 0 & 0 & 0 & 0 & 0 \\ a_t & b_t & c_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_t & b_t & c_t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_r & b_r & c_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_r & b_r & c_r \end{bmatrix}$$

and

$a_r \ b_r \ c_r$ are direction cosines of $\bar{r}$, etc.

c. Limitations

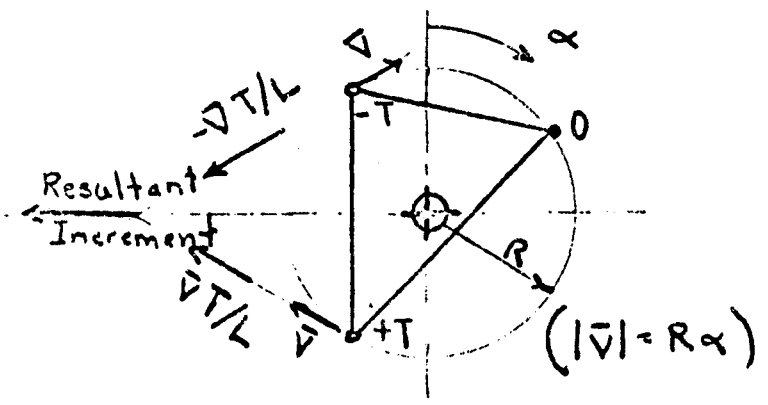
At the beginning of Section 5 it was stated some of the terms to be developed would be applicable only to doubly symmetric beam sections, and that use of a structural model of a beam would disclose such limitations. It is not intended to treat this subject extensively; one example will be chosen as an illustration.

It is familiar knowledge that a beam may have a centroidal axis different from a "shear center" axis. The doubly symmetrical beam avoids this by having a center of symmetry applicable for just about every imaginable property.

Consider a beam with three chords at the apexes of a triangle, subject only to pure moment. Since a beam with three chords is statically determinate, the chord loads are independent of their areas. Thus, it becomes evident that the zeros in Equations 27 and 28 relating ΔM_r to α, will be preserved only if by accident the centroid and a "differential stiffness" center, purely dependent on the geometry of the positions of the chords, coincide. The "centroid" of

the beam and the "centroid" of the triangle coincide if the chord areas are equal, but the "differential stiffness" center is not at the centroid, it is at the center of the circumscribed circle , as may be proved as follows. If the bending moment vector is perpendicular to one of the sides of the triangle defined by the chords as apexes, then the two chords at the ends of that side will have equal and opposite loads and the third chord will be unloaded.

The adjoining sketch shows this situation. Rotation about this center produces a resultant through this same center.

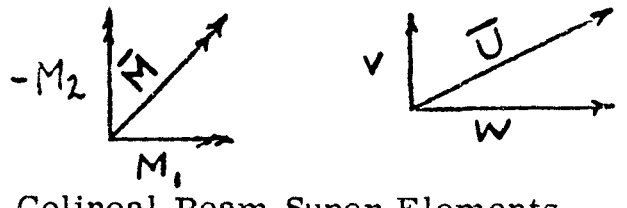


(The center of a circumscribed circle is at the common intersection of the perpendicular bisectors of all three sides.) The centroid and center of circumscribed circle coincide only in an equilateral triangle.

Although a beam evidently must be symmetrical to avoid special corrections it is not necessary that s and t be the principal axes. Consider Equations 33 and 34.

$$\Delta M_{Ar} = M_{2A}(v_A - v_B) + M_{1A}(w_B - w_A)$$

This is a dot product of two vectors, the resultant moment and the resultant displacement (B vs A), and is independent of reference coordinates.



$$\bar{M} \cdot \bar{U} = M_2 v + M_1 w$$

6. Colineal Beam Super Elements

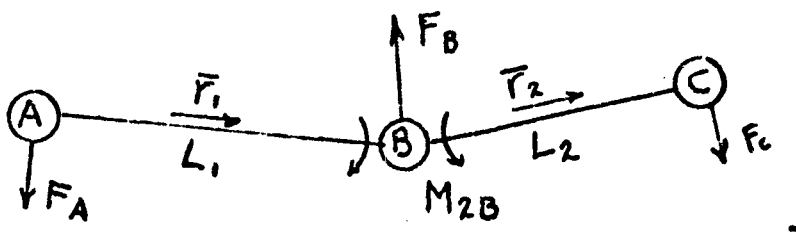
A colineal super element can, when needed, be broken down into a string of segments, each one a beam as described in Section 5. The differential stiffness is then assembled from all these segments. It is

important to note that here must be a full set of torsion elements to accommodate ΔM_r, even though there may be no torsion in the applied loading.

7. Coplanar Beam Super Elements

It is important to notice that the bending moment vector direction at any grid point is uniquely determined by the position of that and the two adjoining grid points.

The development which follows is based on the premise that the curvature induced by loading is small compared with initial curvature of the beam. (The fact that this makes the development unsuitable for slightly curved beams can be gotten around simply by using individual beam segments.) For the beams sufficiently curved (or polygonal) to satisfy these premises, the differential stiffness can be obtained by assuming that when the plane defined by three consecutive grid points rotates the shears and moment rotate with it. Only in-plane bending contributes to differential stiffness. This development preserves an important characteristic of coplanar beams: There are no moment equations at interior grid points, either in the super element or in its differential stiffness matrix.



$\bar{r}_1$ and $\bar{r}_2$ are unit vectors.
Positive values as shown if $\bar{s} = \bar{s}^*$
Positive values opposite if $\bar{s} = -\bar{s}^*$

$$\bar{s}^* = \frac{\bar{r}_1 \times \bar{r}_2}{|\bar{r}_1 \times \bar{r}_2|}$$

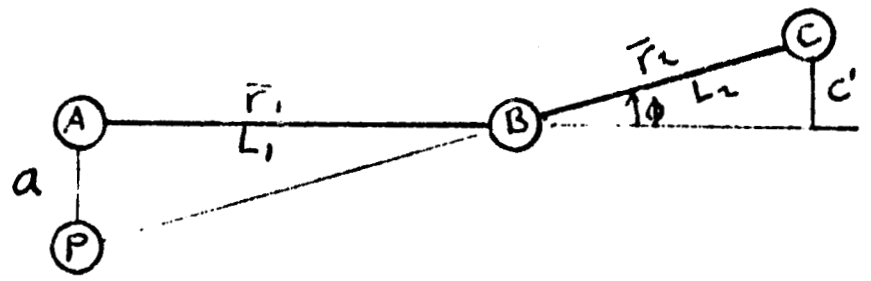
$$\bar{s} = \pm \bar{s}^*$$

Only force and displacement components (F and u) parallel to $\bar{s}$ are considered.

$$\begin{Bmatrix} \Delta F_A^s \\ \Delta F_B^s \\ \Delta F_C^s \end{Bmatrix} = K_s \begin{Bmatrix} u_A^s \\ u_B^s \\ u_C^s \end{Bmatrix} \tag{37}$$

where K_s will now be derived.

We can treat the forces on grid point A as typical. In considering out of plane displacements it is convenient to consider F_A as a vector between A and the auxiliary point P as shown below.



The normal forces at A in terms of normal displacements at A, B and C are given below.

$$\Delta F_A^s = \frac{F_A}{a} \begin{bmatrix} -1 & +\frac{(a+c')}{c'} & -\frac{a}{c'} \end{bmatrix} \begin{Bmatrix} u_A^s \\ u_B^s \\ u_C^s \end{Bmatrix} \tag{38}$$

This can be improved. Noting that

$$F_A = \frac{M_B}{L_1}$$

$$a = L_1 \tan \phi = L_1 \frac{|\bar{r}_1 \times \bar{r}_2|}{\bar{r}_1 \cdot \bar{r}_2}$$

and

$$c' = L_2 \sin \phi = L_2 |\bar{r}_1 \times \bar{r}_2|,$$

the normal forces at A are as follows.

$$\Delta F_A^s = \frac{M_B}{L_1^2} \frac{\bar{r}_1 \cdot \bar{r}_2}{|\bar{r}_1 \times \bar{r}_2|} \left[-1 \;\middle|\; 1 + \frac{L_1}{L_2} \frac{\bar{r}_1 \cdot \bar{r}_2}{} \;\middle|\; \frac{-L_1}{L_2} \frac{}{\bar{r}_1 \cdot \bar{r}_2} \right] \begin{Bmatrix} u_A^s \\ u_B^s \\ u_C^s \end{Bmatrix} \tag{39}$$

Forces at C are obtained in a similar manner.

$$\Delta F_C^s = \frac{M_B}{L_2^2} \frac{r_1 \cdot r_2}{|r_1 \times r_2|} \left[\frac{-L_2}{L_1 \bar{r}_1 \cdot \bar{r}_2} \;\middle|\; 1 + \frac{L_2}{L_1 \bar{r}_1 \cdot \bar{r}_2} \;\middle|\; -1 \right] \begin{Bmatrix} u_A^s \\ u_B^s \\ u_C^s \end{Bmatrix} \tag{40}$$

Forces at B are given by

$$\Delta F_B^s = -(F_A^s + F_C^s) = \frac{M_B}{L_1 L_2 |r_1 \times r_2|} \left[\bar{r}_1 \cdot \bar{r}_2 \frac{L_2}{L_1} + 1 \;\middle|\; -\bar{r}_1 \cdot \bar{r}_2 \left(\frac{L_2}{L_1} + \frac{L_1}{L_2} \right) - 2 \;\middle|\; \bar{r}_1 \cdot \bar{r}_2 \frac{L_1}{L_2} + 1 \right] \begin{Bmatrix} u_A^s \\ u_B^s \\ u_C^s \end{Bmatrix} \tag{41}$$

These terms may be rearranged to give, for K_s in Equation 37,

$$K_s = \frac{M_B}{L_1 L_2 |\bar{r}_1 \times \bar{r}_2|} \left(\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix} - \bar{r}_1 \cdot \bar{r}_2 \begin{bmatrix} L_2/L_1 & -L_2/L_1 & 0 \\ -L_2/L_1 & \frac{L_2}{L_1}\frac{L_1}{L_2} & -L_1/L_2 \\ 0 & -L_1/L_2 & L_1/L_2 \end{bmatrix} \right) \tag{42}$$

Finally Equation 37 may be transformed to basic x, y, z coordinates, yielding

$$\{\Delta F\} = K_{bb} \{u\} \tag{43}$$

where

$$\{\Delta F\} = \{\Delta F_A^x \quad \Delta F_A^y \quad \Delta F_A^z \quad \Delta F_B^x \ \ldots\ldots\ldots\ldots\ldots\ldots \Delta F_C^z\} \qquad (9x1)$$

$$\{u\} = \{u_A^x \quad u_A^y \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ u_C^z\}$$

$$K_{bb} = T_2^T K_s T_2$$

and

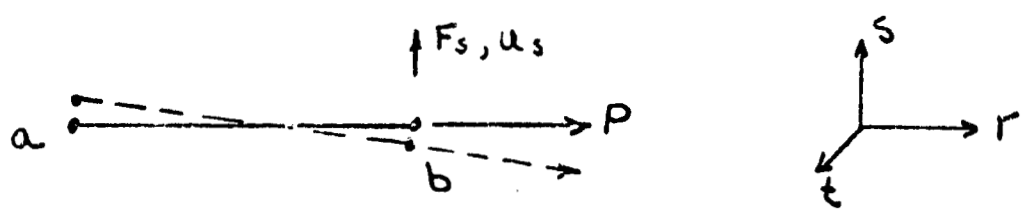
$$T_2 = \begin{bmatrix} a_s & b_s & c_s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_s & b_s & c_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_s & b_s & c_s \end{bmatrix}$$

D. NON GRAVITY LOADS

External loads which remain attached to a grid point and do not rotate nor vary in magnitude produce no incremental loads. When no other kinds of loads act, the differential stiffness matrix is obtained entirely by the formulas in Section C.

Effects of other types of loads produce additional terms (see Equation 2) which in some cases can be evaluated as K_{bc}. K_{bc} will generally be an unsymmetrical matrix. Two illustrative cases are presented.

1. Tangential Load

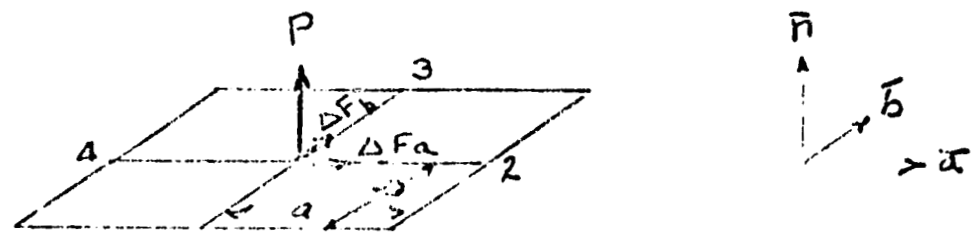


$$\begin{Bmatrix} \Delta F_s \\ \Delta F_t \end{Bmatrix}_b = \frac{P}{L} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_a^s \\ u_b^s \\ u_a^t \\ u_b^t \end{Bmatrix}$$

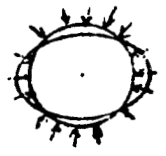
This is typical of rocket thrust.

2. Normal Load

(Illustrated by a regular network)



$$\begin{Bmatrix} \Delta F_a \\ \Delta F_b \end{Bmatrix} = \frac{P}{2} \begin{bmatrix} -1/b & 0 & 1/b & 0 \\ 0 & -1/a & 0 & 1/a \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

This contributes, for example, fifty percent of the driving force for the first buckling mode of a ring loaded by external pressure 

III. PROGRAM FOR THE EIGHTH QUARTER

The following effort is scheduled for the eighth quarter.

1. Final checkout of the NASTRAN system on the IBM S/360 model 50 at Ames Research Center.

2. Conversion of NASTRAN to the UNIVAC 1108.

3. Completion of the Statics Training Manuals and Demonstration Problems.

4. Teaching of the Statics Course at GSFC during May.

5. Initiate implementation of Bell Aerosystems axi-symmetric elements.

6. Continuation of dynamics module checkout including final checkout of complex eigenvalue extraction routine.